SUNY-NTG-95/50

Scalar susceptibility in QCD and the multiflavor Schwinger model

A. Smilga

ITEP, B. Cheremushkinskaya 25, Moscow 117259, Russia

and

J.J.M. Verbaarschot

Department of Physics

SUNY, Stony Brook, New York 11794

Abstract

We evaluate the leading infrared behavior of the scalar susceptibility in QCD and in the multiflavor Schwinger model for small non-zero quark mass m and/or small nonzero temperature as well as the scalar susceptibility for the finite volume QCD partition function. In QCD, it is determined by one-loop chiral perturbation theory, with the result that the leading infrared singularity behaves as $\sim \log m$ at zero temperature and as $\sim T/\sqrt{m}$ at finite temperature. In the Schwinger model with several flavors we use exact results for the scalar correlation function. We find that the Schwinger model has a phase transition at $T = 0$ with critical exponents that satisfy the standard scaling relations. The singular behavior of this model depends on the number of flavors with a scalar susceptibility that behaves as $\sim m^{-2/(N_f+1)}$. At finite volume V we show that the scalar susceptibility is proportional to $1/m^2V$. Recent lattice calculations of this quantity by Karsch and Laermann are discussed.

November 1995

1 Introduction.

The scalar susceptibility in QCD is defined as

$$\chi = \int d^4x \langle \sum_{i=1}^{N_f} \bar{q}_i q_i(x) \sum_{i=1}^{N_f} \bar{q}_i q_i(0) \rangle - V \langle \sum_{i=1}^{N_f} \bar{q}_i q_i \rangle^2 = \frac{1}{V}\, \partial_m^2 \log Z \Big|_{m=0}, \tag{1.1}$$

where V is the four dimensional Euclidean volume and the averaging is performed either over the vacuum state or over the thermal ensemble. (In the Euclidean approach the latter corresponds to an asymmetric box with imaginary time extension of $\beta = 1/T \ll L$, i.e. $V = L^3\beta$). The definition (1.1) is for a diagonal mass matrix with equal quark masses. It is especially interesting to study this quantity in the neighborhood of the thermal phase transition point. It is expected that for QCD with two massless flavors a second order phase transition occurs leading to restoration of chiral symmetry [1] (see [2, 3] for recent reviews) with a diverging susceptibility. This can be understood simply in terms of Landau mean field theory. For a system with order parameter η coupled to external field h with a second order phase transition at $T = T_c$, the fluctuations of the order parameter are described by the effective potential

$$V^{\rm eff}(\eta) = A(T - T_c)\eta^2 + B\eta^4 + C\eta h. \tag{1.2}$$

In QCD η is the chiral condensate and h is the quark mass. At $T = T_c$, the minimum of the potential occurs at $\eta \sim h^{1/3}$ which gives the law $\chi = \partial\eta/\partial h \sim h^{-2/3}$ for the susceptibility. On the other hand if T is close to T_c but $T \neq T_c$ and the external field is weak enough

$$h \ll |T - T_c|^{3/2}, \tag{1.3}$$

the quartic term in (1.2) is irrelevant and the scaling law is $\chi \sim |T - T_c|^{-1}$ both above and below T_c (the proportionality constant in these two regions differs by a factor 2).

Recently, on the basis of lattice simulations of the 3-dimensional Gross-Neveu model it has been suggested that a second order phase transition involving soft modes consisting out of fermions has critical exponents given by mean field theory [4]. In particular, if this is also true for QCD with two massless flavors, we get

$$\langle \bar{q}q \rangle \sim m^{\frac{1}{3}}, \tag{1.4}$$

$$\chi \sim m^{-\frac{2}{3}}, \tag{1.5}$$

at $T = T_c$, and

$$\chi \sim \frac{1}{|T - T_c|} \tag{1.6}$$

at

$$\Lambda^{\frac{1}{3}} m^{\frac{2}{3}} \ll |T - T_c| \ll T_c, \tag{1.7}$$

where Λ is a typical hadronic mass scale. These scaling laws have been reproduced by a simple stochastic matrix model [5, 6]. The scalar susceptibility was recently measured for lattice QCD with two light flavors [7]. They found a diverging susceptibility at $T = T_c$ with critical exponent $\delta^{-1} = 0.24 \pm 0.03$ which does not agree with the prediction $\delta^{-1} = 1/3$ of the Landau mean field theory. Obviously, further numerical measurements of the critical indices in QCD are highly desirable.

In this paper we will study the scalar susceptibility in three different cases. First, in section 2 the scalar susceptibility is evaluated for the multi-flavor massive Schwinger model which shares many common qualitative features with QCD and, in particular, it shows a phase transition with "restoration" of chiral symmetry at $T = 0$. All other critical exponents that can be defined in the SM will be evaluated as well, and it will be shown that they satisfy the scaling relations modified for a phase transition at zero temperature. Second, we evaluate the scalar susceptibility in QCD at low temperatures using chiral perturbation theory. Third, in order to estimate finite size effects in lattice calculations, we calculate the scalar susceptibility in volumes with spatial length below the Compton wave length of the pion.

a b

Fig. 1. Connected (a) and disconnected (b) graphs contributing to the scalar susceptibility of quarks propagating in a background gluon field.

Before proceeding further, we should note that the scalar susceptibility as defined in (1.1) involves a quadratic ultraviolet divergence due to a trivial perturbative graph

depicted in Fig. 1a. The situation is the same as with the chiral condensate which involves a trivial divergent perturbative contribution $\sim m\Lambda_{UV}^2$. The predictions (1.4,1.5, 1.6) hold for the infrared sensitive part of $\langle\bar{q}q\rangle$ and χ.

The susceptibility (1.1) can be written as the sum of a connected and a disconnected contribution corresponding to the graphs of Figs. 1a and 1b, respectively,

$$\chi = N_f \chi^{\rm con} + N_f^2 \chi^{\rm dis} \tag{1.8}$$

The connected contribution to the susceptibility was calculated in [8]. The disconnected contribution is defined by

$$\chi^{\rm dis} = \int d^4x \langle \bar{u}u(x)\bar{d}d(0)\rangle - V\langle\bar{u}u\rangle\langle\bar{d}d\rangle = \frac{1}{V}\partial_{m_u}\partial_{m_d}\log Z, \tag{1.9}$$

where m_u and m_d are two different quark masses that are put to zero after differentiation. We will obtain the latter contribution from the difference of χ and $\chi^{\rm con}$.

Both scalar susceptibilities can be expressed in terms of the eigenvalues λ_k of the Dirac operator

$$\chi^{\rm dis} = \frac{1}{V}\left[\left\langle\left(\sum_k \frac{1}{i\lambda_k + m}\right)^2\right\rangle - \left\langle\sum_k \frac{1}{i\lambda_k + m}\right\rangle^2\right], \tag{1.10}$$

$$\chi^{\rm con} = -\frac{1}{V}\left\langle\sum_k \frac{1}{(i\lambda_k + m)^2}\right\rangle. \tag{1.11}$$

A related susceptibility is the pseudo-scalar susceptibility which in terms of the Dirac eigenvalues is given by

$$\chi^{\pi} = \frac{1}{V}\left\langle\sum_k \frac{1}{\lambda_k^2 + m^2}\right\rangle. \tag{1.12}$$

2 The Schwinger model

In this section we discuss the Schwinger model (SM) with N_f light flavors, and determine the critical exponents from known results for the correlation functions. For the critical exponents we will follow the convention of [9].

Critical behavior shows up only in the SM with several flavors. In the standard Schwinger model ($N_f = 1$), there is no non-anomalous global symmetry to be broken spontaneously and no reason for the phase transition to occur. The scalar susceptibility in the standard SM was determined recently in [10]. It is just a finite constant.

The theory with $N_f > 1$ and zero fermion masses has the global symmetry $SU_L(N_f) \times SU_R(N_f)$ (much like in QCD) and the potential possibility of its spontaneous breaking with generation of fermion condensate exists. Coleman's theorem [11] prevents, however, spontaneous breaking of a continuous symmetry in 2 dimensions. So, a QCD–like phase transition cannot occur in a 2D theory at a nonzero temperature. Nevertheless, it turns out that the dynamics of the SM at small T is similar to that of a theory with second order phase transition in the region of temperatures slightly above critical. One can say that the phase transition *does* occur at $T = 0$.

The Lagrangian of the model is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + i\sum_{f=1}^{N_f} \bar{q}_f \gamma_\mu(\partial_\mu - igA_\mu)q_f - m\sum_{f=1}^{N_f} \bar{q}_f q_f, \tag{2.1}$$

where g is the coupling constant with the dimension of mass, and, for simplicity, we assumed that all quark masses are equal.

Let us study this model in the region $m \ll g$. The particle spectrum involves a massive photon [12] with mass

$$\mu_+ \sim g\sqrt{\frac{N_f}{\pi}} + O(m) \tag{2.2}$$

and 'quasi-Goldstone' particles[1] with the mass [13, 14]

$$\mu_- \sim g^{\frac{1}{N_f+1}} m^{\frac{N_f}{N_f+1}}. \tag{2.3}$$

This gives us the critical exponent $\mu = N_f/(N_f + 1)$.

At nonzero m, the chiral $SU_L(N_f) \times SU_R(N_f)$ symmetry of the massless SM Lagrangian is broken explicitly, and the formation of the chiral condensate becomes possible. The chiral condensate involves a UV divergent piece

$$\langle \bar{q}q \rangle \sim m \log \Lambda_{UV}, \tag{2.4}$$

[1]It is better to use the term 'quasi-Goldstone' than the term 'pseudo-Goldstone' commonly used for pions because 'quasi-Goldstone' states in the SM become sterile in the chiral limit $m \to 0$. That conforms with Coleman's theorem which forbids the existence of massless interacting particles in 2 dimensions.

and an infrared contribution (sensitive to the small eigenvalues of the Euclidean Dirac operator). The latter has been determined in [14, 15] with the result

$$\langle \bar{q}q \rangle \sim m^{\frac{N_f-1}{N_f+1}} g^{\frac{2}{N_f+1}}, \tag{2.5}$$

providing us the critical exponent $\delta = (N_f + 1)/(N_f - 1)$. The susceptibility is

$$\chi = \frac{\partial \langle \bar{q}q \rangle_m}{\partial m} \sim \left(\frac{g}{m}\right)^{2/(N_f+1)} \tag{2.6}$$

In the region $\mu_+^{-1} \ll |x| \ll \mu_-^{-1}$ the vacuum scalar correlator is given by [12, 16, 14]

$$\langle \bar{q}q(x)\bar{q}q(0) \rangle_0 \sim \frac{g^{2/N_f}}{|x|^{2-2/N_f}}. \tag{2.7}$$

The associated critical exponent is $\zeta = 2 - 2/N_f$ (To make contact with the standard theory of phase transitions, x should be assumed space-like, but the behavior (2.5) holds of course for any x due to Lorentz-invariance). At $|x| \gg \mu_-^{-1}$ the correlator levels off at the value of the square of the chiral condensate (2.5).

In the region $\mu_- \ll T \ll \mu_+$ (weak field limit) the condensate is given by [15]

$$\langle \bar{q}q \rangle \sim m \left(\frac{g}{T}\right)^{2/N_f}, \tag{2.8}$$

which leads to the susceptibility

$$\chi \sim \left(\frac{g}{T}\right)^{2/N_f}, \tag{2.9}$$

and the critical exponent $\gamma = 2/N_f$.

The correlation length in this case is just inverse fermion Matsubara frequency $\sim 1/T$ which gives the critical exponent $\nu = 1$. For $\mu_- \ll T$ the energy density is that of a gas of massless particles. Therefore, $\epsilon \sim \frac{\pi}{3}T^2$ and the specific heat for zero field is given by

$$C = \frac{d\epsilon}{dT} \sim T \tag{2.10}$$

leading to $\alpha = -1$.

The above results for the critical exponent have been summarized in Table 1. We also show the critical exponents for mean field theory. Note that for $N_f = 2$ *some* (namely, δ and γ), but not all of the exponents coincide with the predictions of MFT.

exponent	MFT	SM
α	0	-1
β	$\frac{1}{2}$	—
γ	1	$\frac{2}{N_f}$
δ	3	$\frac{N_f+1}{N_f-1}$
ε	0	—
μ	$\frac{1}{3}$	$\frac{N_f}{N_f+1}$
ν	$\frac{1}{2}$	1
ζ	0	$2-\frac{2}{N_f}$

Table 1: The critical exponents for mean field theory (MFT) and the Schwinger model (SM). Conventions are as in Landau and Lifshitz [5].

It is instructive to write down an effective non-local lagrangian from where these values of critical exponents can be inferred:

$$\mathcal{L} = A\eta(\Delta + BT^2)^{1/N_f}\eta + C\eta^{\frac{2N_f}{N_f-1}} + D\eta h \tag{2.11}$$

Of course, (2.11) is just a shorthand for the values of critical indices obtained and should be understood as such.

In standard theory of second order phase transitions with a nonzero critical temperature the above 8 critical exponents satisfy 5 universal thermodynamic relations and the hyper-scaling relation. The fact that the critical temperature is zero brings about a number of modifications compared to the standard theory of phase transitions:

- The critical exponent β refers to the broken phase and therefore cannot be defined.
- In the strong field limit $h \gg t^{\nu/\mu}$ the partition function depends on the temperature as $\propto \exp\{-\frac{h^{\mu/\nu}}{t}\}$. Then the critical exponent ε becomes singular for $t \to 0$ and cannot be defined. Specifically, for the SM the free energy of the gas of bosons with the mass of order μ_- remains exponentially small until $t \sim \mu_- \propto m^\mu$ ($\nu = 1$ in our case).
- Because $c_p = -T\partial^2\Phi/\partial T^2$ and $T \equiv T - T_c$ (with $T_c = 0$), an extra power of T emerges leading to the Rushbrooke scaling relation

$$\alpha + 2\beta + \gamma = 1 \tag{2.12}$$

instead of 2. After elimination of β and ϵ we have only 3 instead of the usual 5 relations:

$$\alpha + \frac{\gamma(\delta+1)}{\delta-1} = 1 \tag{2.13}$$

$$\nu(2-\zeta) = \gamma. \tag{2.14}$$

$$\mu(1+\gamma-\alpha) = 2\nu \tag{2.15}$$

- The hyperscaling relation follows from the condition that the total free energy is of the order of TV/ξ^d, where V is the spatial volume of dimension d and ξ is the correlation length. For $T_c = 0$ we obtain

$$\nu d = -\alpha \tag{2.16}$$

instead of $2-\alpha$.

The meaning of the hyperscaling relation is that loop corrections to the Green's functions estimated from the effective lagrangian (2.11) are of the same order, as far as powers are concerned, as the tree expressions. We emphasize again that one cannot perform *serious* loop calculations with the effective lagrangian (2.11). If trying to do so, logarithmic factors would appear which would shift "tree level" values of exponents.

3 QCD at low temperatures

The primary interest of the quantity (1.1) is that its critical behavior can provide information on the physics of the phase transition in QCD. However, our second remark is that the leading infrared behavior of χ can be determined *exactly* in the low temperature region. The proper technique to extract it is chiral perturbation theory [17].

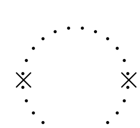

Fig. 2. Pseudo-Goldstone loop determining the scalar susceptibility.

The leading infrared behaviour is determined by the graph in Fig. 2 involving a loop of quasi-massless pseudo-Goldstone bosons. The effective low-energy Lagrangian of QCD has the form

$$\mathcal{L}^{\rm eff} = \frac{1}{4}F_\pi^2 {\rm Tr}(\partial_\mu U^\dagger)(\partial_\mu U) + \Sigma {\rm Re Tr}\{\mathcal{M}U\} + \cdots, \tag{3.1}$$

where $U = \exp\{2i\phi^a t^a/F_\pi\}$ and ϕ^a are the pseudo-Goldstone fields. The chiral condensate is denoted by $\Sigma = |\langle \bar{q}q\rangle_0|$ (no summing over colors assumed), and $\mathcal{M}$ is the quark mass matrix. The partition function with $\mathcal{M} = {\rm diag}(m, m, \cdots, m)$ has been calculated by Gasser and Leutwyler [18]. Their expression for the free energy density of lukewarm pion gas is

$$f = \epsilon_0(M_\pi) - \frac{N_f^2 - 1}{2} g_0(T, M_\pi) + \frac{N_f^2 - 1}{4N_f}\,\frac{M_\pi^2}{F_\pi^2} g_1^2(T, M_\pi) \tag{3.2}$$

where

$$g_0(T, M_\pi) = -\frac{T}{\pi^2}\int p^2 dp \ln[1 - \exp(-E/T)], \tag{3.3}$$

$E = \sqrt{p^2 + M_\pi^2}$, and

$$g_1(T, M_\pi) \;=\; \frac{1}{2\pi^2}\int \frac{p^2 dp}{E[\exp(E/T) - 1]} = -\frac{\partial g_0(T, M_\pi)}{\partial M_\pi^2} \tag{3.4}$$

Here, M_π stands for the common mass of the $N_f^2 - 1$ pseudo-Goldstone modes given by the Gellmann-Oakes-Renner relation $F_\pi^2 M_\pi^2 = 2m\Sigma$.

Substituting in Eq.(3.2) the expansion [19, 20]

$$g_0 = \frac{\pi^2}{45}T^4 - \frac{T^2 M_\pi^2}{12} + \frac{T M_\pi^3}{6\pi}, \tag{3.5}$$

we obtain

$$\begin{aligned} f \;=\; & -\frac{\pi^2}{90}(N_f^2 - 1)T^4 - \frac{N_f}{2}F_\pi^2 M_\pi^2 \left(1 - \frac{N_f^2 - 1}{12N_f}\frac{T^2}{F_\pi^2} - \frac{N_f^2 - 1}{288N_f^2}\frac{T^4}{F_\pi^4}\right) \\ & - \frac{N_f^2 - 1}{12\pi} T M_\pi^3 \left(1 + \frac{1}{8N_f}\frac{T^2}{F_\pi^2}\right) \\ & - \frac{N_f^2 - 1}{64\pi^2} M_\pi^4 \log\left(\frac{\Lambda^2}{M_\pi^2}\right), \end{aligned} \tag{3.6}$$

Consider first the case of zero temperature. For the infrared singular contribution to the scalar susceptibility we obtain

$$\chi^{IR} = \frac{N_f^2 - 1}{8\pi^2} \left(\frac{\Sigma}{F_\pi^2}\right)^2 \log \frac{\Lambda^2}{M_\pi^2}, \tag{3.7}$$

which was first obtained in [21] from an analysis of the spectral function in the scalar channel and the PCAC hypothesis.

The infrared singular connected contribution to the susceptibility was calculated in [8] (One should just substitute 1 for $\mathrm{Tr}\{t^a t^b\} = \delta^{ab}/2$ in Eq.(2.13) of Ref.[8].). The result is

$$\chi^{\mathrm{IR\,con}} = \frac{N_f^2 - 4}{16\pi^2 N_f} \left(\frac{\Sigma}{F_\pi^2}\right)^2 \log \frac{\Lambda^2}{M_\pi^2}. \tag{3.8}$$

Using (1.8) we immediately obtain the disconnected contribution

$$\chi^{\mathrm{IR\,dis}} = \frac{N_f^2 + 2}{16\pi^2 N_f^2} \left(\frac{\Sigma}{F_\pi^2}\right)^2 \log \frac{\Lambda^2}{M_\pi^2}. \tag{3.9}$$

Instead of using the partition function of Gasser and Leutwyler we can calculate the susceptibility also using the same technique as in [8]. Choosing $\mathcal{M} = \mathrm{diag}(m, m, \cdots, m)$, one immediately obtains the vertex

$$\langle 0| \sum_f \bar{q}_f q_f |\phi^a \phi^b\rangle = \frac{2\Sigma}{F_\pi^2}\delta^{ab}, \tag{3.10}$$

which, by evaluation of the diagram in Fig. 2, reproduces the result (3.7).

The infrared singular contribution to the susceptibility af finite temperature can be obtained directly from (3.6) as well

$$\chi_T^{IR} = \frac{(N_f^2 - 1)}{4\pi} \frac{T}{\sqrt{2m}} \left(\frac{\Sigma}{F_\pi^2}\right)^{3/2}, \tag{3.11}$$

where the Gellmann-Oakes-Renner relation has been used. The connected contribution to the susceptibility was not calculated in [8]. However, the zero temperature result of Ref.[8] can be extended immediately to finite T by making the substitution

$$\int \frac{d^4 p}{(2\pi)^4} \to T \sum_n \int \frac{d^3 p}{(2\pi)^3}, \tag{3.12}$$

where the sum is over all Matsubara frequencies ($p_0 = 2\pi n/\beta$). The infrared singular part comes only from term $n = 0$ and the result is

$$\chi_T^{\mathrm{IR\,con}} = \frac{N_f^2 - 4}{8\pi N_f} \frac{T}{\sqrt{2m}} \left(\frac{\Sigma}{F_\pi^2}\right)^{3/2}. \tag{3.13}$$

This implies that at finite temperature the spectrum of the Dirac operator is nonanalytic for small eigenvalues. The disconnected contribution is

$$\chi_T^{\text{IR dis}} = \frac{N_f^2+2}{8\pi N_f^2}\frac{T}{\sqrt{2m}}\left(\frac{\Sigma}{F_\pi^2}\right)^{3/2}. \tag{3.14}$$

The relations (3.11, 3.13, 3.14) are quite analogous to the relations for the magnetic susceptibility for ferromagnets known for a long time. They are also determined by a loop of pseudo-Goldstone particles (the magnons) depicted in Fig. 2 and have the same behavior[2] [23]

$$\chi^{\text{magnon}} = \left.\frac{\partial M}{\partial H}\right|_{H=0} \sim \frac{T}{\sqrt{H}}. \tag{3.15}$$

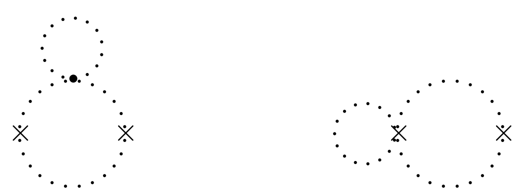

Fig. 3. Two-loop contributions to the scalar susceptibility: a) mass renormalization; b) vertex renormalization.

The relations (3.11, 3.13, 3.14) hold in the low-temperature region

$$M_\pi \ll T \ll T_c. \tag{3.16}$$

This makes a direct comparison with recent lattice calculations [7] impossible. At not so low temperatures, besides the graph in Fig. 2, also higher order graphs in chiral perturbation theory contribute. The relevant two-loop graphs are depicted in Fig. 3, but the results for the susceptibility can be obtained directly from the partition function as well (3.6),

$$\chi_T^{IR} = \frac{(N_f^2-1)}{4\pi}\frac{T}{\sqrt{2m}}\left(\frac{\Sigma}{F_\pi^2}\right)^{3/2}\left(1+\frac{1}{8N_f}\frac{T^2}{F_\pi^2}\right). \tag{3.17}$$

[2]Recently, a full nonlinear effective Lagrangian has been constructed [22] in a way which makes the analogy between the theory of ferromagnets and CPT most transparent.

The two loop temperature dependence can be absorbed in the one-loop temperature modification of the condensate and the pion decay constant as obtained in [18]

$$\Sigma(T) = \Sigma(0)\left(1-\frac{N_f^2-1}{12N_f}\frac{T^2}{F_\pi^2}\right), \tag{3.18}$$

$$F_\pi(T) = F_\pi(0)\left(1-\frac{N_f}{24}\frac{T^2}{F_\pi^2}\right). \tag{3.19}$$

It is not clear what happens to next order in T. Temperature corrections to the condensate have been found to three-loop level in [20]. Two-loop effects in $M_\pi^2(T)$ have been extracted in [24, 25], but $F_\pi^2(T)$ at the two-loop level is currently unknown. It is not clear whether one can just substitute the temperature dependent Σ and F_π^2 in (3.11) to all orders in the temperature expansion. The same question can be asked concerning the Gellmann-Oakes-Renner relation. Taking into account corrections of order T^2/F_π^2, it also holds at nonzero temperatures. Whether it holds at higher orders is an open question [26].

4 The scalar susceptibility in finite volumes

The results (3.7, 3.11, 3.13, 3.14) are valid in the thermodynamic limit which means that the spatial length of the box where the theory is defined is much larger than the pion Compton wave length. However, realistic boxes used in lattice calculations can never be made so large if the pion mass is small enough (which is in turn necessary for the quantitative analytic predictions to be possible). In this section we consider the opposite limit

$$\Lambda^{-1} \ll L \ll M_\pi^{-1} \sim \frac{1}{\sqrt{m\Lambda}} \tag{4.1}$$

where the susceptibility can be found from the exact results for the finite-volume partition function of [27]. In this range the susceptibility is expected to be determined by finite volume effects and be of order

$$\frac{\Sigma}{m}\left(a_0 + a_1\frac{1}{mV\Sigma} + O\left(\frac{1}{(mV\Sigma)^2}\right)\right) \tag{4.2}$$

which is much larger than its thermodynamic limit $\sim \Lambda^2$. If $a_0 \neq 0$, the result becomes independent of the volume suggesting that it holds for $V \to \infty$ outside of the range (4.1) provided that we are sufficiently close to the chiral limit so that $\Sigma/m \gg \Lambda^2$. This is indeed what happens for the pion susceptibility but not for the scalar susceptibility (see below). In general, the results of this section have the same status as in Ref.[27]. They say little about the properties of the theory in the physical infinite volume limit but can be used to test the validity of numerical calculations in QCD which are performed for finite volumes.

The theoretically simplest susceptibility is the pseudoscalar (or pion) susceptibility. Using the Banks-Casher relation [28], it can be related immediately to the chiral condensate, (see eq. (1.12))

$$\chi^\pi = \frac{\Sigma}{m}, \tag{4.3}$$

in the limit that $\Sigma m V \gg 1$. The validity of this result extends into the thermodynamic limit outside of the range (4.1).

As follows immediately from (1.11,1.10) $\chi^{\rm con}$ can be calculated from average spectral density

$$\rho(\lambda) = \langle \omega(\lambda, A) \rangle = \langle \frac{1}{V} \sum_n \delta(\lambda - \lambda_n) \rangle \tag{4.4}$$

On the other hand, $\chi^{\rm dis}$ can be expressed into the connected two-point level correlation function

$$\rho_c(\lambda, \lambda') = V \left[\langle \omega(\lambda) \omega(\lambda') \rangle - \rho(\lambda)\rho(\lambda') \right]. \tag{4.5}$$

If, apart from the pairing $\pm\lambda_k$, the eigenvalues of the Dirac operator are uncorrelated [29], i.e. if $< f(\lambda_n) g(\lambda_m) >=< f(\lambda_n) >< g(\lambda_m) >$ for $n \neq m$ and λ_n, $\lambda_m > 0$, we have in the limit of large V and for positive λ, λ'

$$\rho_c(\lambda, \lambda') = \rho(\lambda) \delta(\lambda - \lambda'). \tag{4.6}$$

Using the $U_A(1)$ symmetry of the Dirac spectrum the disconnected susceptibility (1.10) can be written as

$$\chi^{\rm dis} = \int_0^\infty d\lambda \int_0^\infty d\lambda' \frac{4m^2 \rho_c(\lambda, \lambda')}{(\lambda^2 + m^2)(\lambda'^2 + m^2)} \tag{4.7}$$

which after insertion of the correlation function (4.6) leads to

$$\chi^{\mathrm{IR\,dis}} = \frac{\pi\rho(0)}{m} = \frac{\Sigma}{m}, \tag{4.8}$$

where the second equality is the Banks-Casher relation [28].

Because the diagonalization of the Dirac operator induces correlations between the eigenvalues, we expect quite a different prediction from the finite volume partition function.

For $N_f = 2$ the finite volume partition function is known [27]. For $\theta = 0$,

$$Z = \frac{2}{V\Sigma(m_u + m_d)} I_1(V\Sigma(m_u + m_d)) \tag{4.9}$$

where $V = L^3/T$ in the 4D Euclidean volume. The susceptibilities χ^{con} and χ^{dis} can be disentangled by differentiating with respect to *different* quark masses. Because the partition function depends on the quark masses via the sum $m_u + m_d$ we find

$$\chi^{\mathrm{con}} = 0. \tag{4.10}$$

In the limit $\kappa \equiv V\Sigma m \gg 1$ the disconnected contribution simplifies

$$\chi^{\mathrm{dis}} = \frac{3}{8m^2 V}. \tag{4.11}$$

In present day lattices the zero mode states are mixed with the much larger number of nonzero mode states. Therefore, the partition function is effectively calculated in the sector $\nu = 0$. Existing numerical calculations in the instanton liquid model [30] are also done for $\nu = 0$. In a sector with fixed topological charge the finite volume partition function has been calculated analytically for an arbitrary number of flavors with equal mass [27]. To leading order in κ^{-1}, the partition function in the sector of zero topological charge is given by

$$Z^{\mathrm{eff}}_{\nu=0} \sim \frac{\exp(N_f\kappa)}{\kappa^{N_f^2/2}}. \tag{4.12}$$

For $\kappa \gg 1$ we find

$$N_f\chi^{\mathrm{con}} + N_f^2\chi^{\mathrm{dis}} = \frac{N_f^2}{2}\frac{1}{m^2 V}. \tag{4.13}$$

In general we have not been able to calculate the connected and disconnected contributions to the susceptibility separately. However, for $N_f = 2$ the partition function is known for different quark masses [27]. For $\nu = 0$ we find

$$Z_{\nu=0} = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{2I_0(V\Sigma(m_u^2 + m_d^2 + 2m_u m_d \cos\theta)^{1/2})}{V\Sigma(m_u^2 + m_d^2 + 2m_u m_d \cos\theta)^{1/2}}, \tag{4.14}$$

which allows us to calculate the connected and disconnected pieces of the susceptibility separately. The result for $\kappa \gg 1$ is $\chi^{\rm con} = 1/2m^2V$ and $\chi^{\rm dis} = 1/4m^2V$. For $N_f = 0$ the two contributions to the susceptibility can be obtained from the spectral density and the two level spectral correlation function which can be derived from chiral random matrix theory. The result for $\kappa \gg 1$ is [31] $\chi^{\rm con} = 0$ and $\chi^{\rm dis} = 1/4m^2V$. Our conjecture for arbitrary N_f and $\nu = 0$ consistent with all the above results is

$$\chi^{\rm con} = \frac{N_f}{4} \frac{1}{m^2V} F^{\rm con}(\kappa), \tag{4.15}$$

$$\chi^{\rm dis} = \frac{1}{4m^2V} F^{\rm dis}(\kappa), \tag{4.16}$$

in agreement with the simplest possible flavor dependence consistent with (4.13). Both $F^{\rm con}(\kappa)$ and $F^{\rm dis}(\kappa)$ approach 1 for $\kappa \gg 1$, but will in general depend on N_f for finite κ. The disconnected contribution is suppressed by a factor $1/mV\Sigma$ with respect to the result for uncorrelated eigenvalues. The coefficient a_0 in (4.2) turns out to be zero which is in agreement with the fact that there are no massless scalar particles.

The quark mass dependence of the scalar susceptibility has been calculated by lattice QCD simulations only for relatively large quark masses [7]. In this work with $m^2V \approx 1$ (in units of the lattice spacing) a quark mass dependence of $\sim 1/m$ is not only found at the critical point but also at lower temperatures where the susceptibility levels off at a significantly lower value. This result is in between the finite volume prediction (4.13) and the result from chiral perturbation theory (3.11) with mass dependence of $1/m^2$ and $1/\sqrt{m}$, respectively. The lattice results for $\chi^{\rm dis}$ agree with the mass dependence (4.8) suggesting that the eigenvalues of the Dirac operator are only weakly correlated. Two different types of correlations can be considered, namely correlations between eigenvalues corresponding to different gauge field configurations and correlations between eigenvalues obtained from the same lattice gauge configuration. It has been shown [32] that lattice eigenvalues show strong spectral correlations (the latter type), but this does not exclude

the possibility that correlations between eigenvalues corresponding to different members of the ensemble are absent. In random matrix theory it has been shown [33] that spectral averages and ensemble averages are the same. The exciting possibility that this type of generalized ergodicity does not hold for the lattice Dirac eigenvalues deserves further attention.

Acknowledgements

The reported work was partially supported by the US DOE grant DE-FG-88ER40388 and the INTAS grant 93-0283. We are much indebted to Y. Hosotani, A. Larkin, V. Lebedev, and E. Shuryak for illuminating discussions and to H. Leutwyler for helpful correspondence. One of us (A.V.S) acknowledges warm hospitality extended to him by the Nuclear Theory Group of the State University of New York at Stony Brook were this work was done.

References

[1] R.D. Pisarski and F. Wilczek, Phys. Rev. **D29** (1984) 338.

[2] A.V. Smilga, Lectures at the International School of Physics "Enrico Fermi" (Varenna, July 1995), Stony Brook preprint SUNY-NTG/95-34, hep-ph/9508305.

[3] C. DeTar, *Quark gluon plasma in numerical simulations of lattice QCD*, to appear in Quark Gluon Plasma II, R. Hwa ed., World Scientific 1995.

[4] A. Kocic and J. Kogut, Phys. Rev. Lett. **74** (1995) 3109; *Phase transitions at finite temperature and dimensional reduction for fermions and boson*, University of Illinois preprint, 1995 (hep-lat/9507012).

[5] A.D. Jackson and J.J.M. Verbaarschot, *A Random Matrix Model for Chiral Symmetry Breaking*, Stony Brook preprint SUNY-NTG-95/26, 1995.

[6] T. Wettig, A. Schäfer and H.A. Weidenmüller, *The chiral phase transition in a random matrix model with molecular correlations*, Phys. Lett. **B** (1995) (in press).

[7] F. Karsch and E. Laermann, Phys. Rev. **D50** (1994) 6954.

[8] A.V. Smilga and J. Stern, Phys. Lett. **B318** (1993) 531.

[9] L.D. Landau and E.M. Lifshitz, *Statistical Physics, Part I*, Pergamon Press, 1980.

[10] C. Adam, Phys. Lett. **B363** (1995) 79.

[11] S. Coleman, Comm. Math. Phys. **31** (1973) 259.

[12] G. Segrè and W.I. Weisberger, Phys. Rev. **D10** (1974) 1767.

[13] S. Coleman, Ann. Phys. **101** (1976) 239.

[14] A.V. Smilga, Phys. Lett. **B278** (1992) 371.

[15] J.E. Hetrick, Y. Hosotani and S. Iso, *The Massive Multi-flavor Schwinger Model*, University of Minnesota preprint UMN-TH-1324/95.

[16] A. Abada and R.E. Shrock, Phys. Lett. **B267** (1991) 282; A. Abada, Ph.D. Thesis of SUNY Stony Brook, 1992.

[17] See e.g. H. Leutwyler, *Chiral Effective Lagrangians* in: Perspectives in the Standard Model, Proceedings of the 1991 Advanced Study Institute in Elementary Particle Physics, World Scientific 1992.

[18] J. Gasser and H. Leutwyler, Phys. Lett. **B184** (1987) 83.

[19] H.E. Haber and H.A. Weldon, J. Math. Phys. **23** (1981) 1852.

[20] P. Gerber and H. Leutwyler, Nucl. Phys. **B321** (1989) 387.

[21] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl.Phys. **B191** (1981) 301.

[22] H. Leutwyler, Phys. Rev. **D49** (1994) 3033.

[23] E.M. Lifshitz and L.P.Pitaevski, *Statistical Physics, Part II*, p. 299, Pergamon Press, 1980.

[24] J.L. Goity and H. Leutwyler, Phys. Lett. **B228** (1989) 517.

[25] A. Schenk, Nucl. Phys. **B363** (1991) 97; Phys. Rev. **D47** (1993) 5138.

[26] H. Leutwyler, private communication.

[27] H. Leutwyler and A. Smilga, Phys. Rev. **D46** (1992) 5607.

[28] T. Banks and A. Casher, Nucl. Phys. **B169** (1980) 103.

[29] M.A. Nowak, J.J.M. Verbaarschot and I. Zahed, Nucl. Phys. **B324** (1989) 1.

[30] E.V. Shuryak and J.J.M. Verbaarschot, Nucl. Phys. **B341** (1990) 1.

[31] J.J.M. Verbaarschot, *Universal scaling of the valence quark mass dependence of the chiral condensate*, Phys. Lett. **B** (1995) (in press).

[32] M.A. Halasz and J.J.M. Verbaarschot, Phys. Rev. Lett. **74** (1995) 3920.

[33] A. Pandey, Ann. Phys. **119** (1978) 170.